Arithmetic made easy

Our system of numbers is not necessarily the best in that it does not make arithmetic as easy as possible. Surprising as it may seem there is a better way of handling numbers, although the difficulties of re-educating the world probably preclude its use

Dylan Morgan We are very proud of our numbers. Most books on the history of mathematics have a smug section on the inadequacy of Babylonian, Greek and Roman numbers. We can afford this sense of superiority because of course we use the *best* method. But do we?

The question is reasonable: after all, it is one of the most conspicuous facts in the history of science that each generation imagines that its knowledge is very nearly perfect. And each generation (except our own of course) has been shown to be blind to its own inadequacies to an amazing extent. So could it be that a similar blindness is preventing us from seeing room for improvement in so basic a thing as elementary arithmetic?

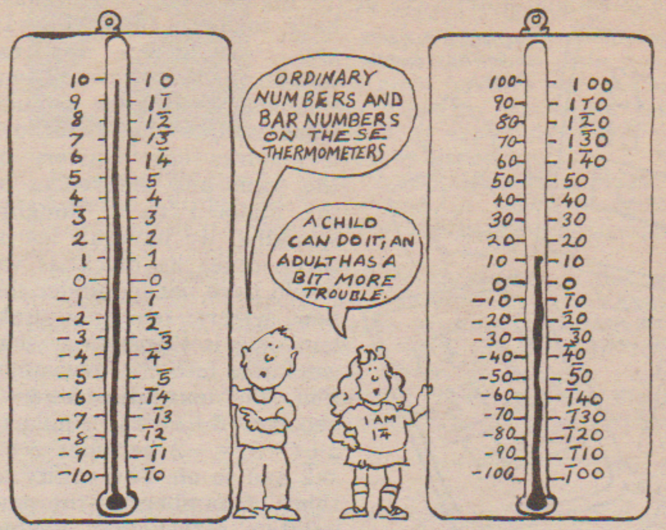
But what do we mean by "good" or "bad" in the context of handling numbers? It is not a question of right or wrong. The Roman who (correctly) multiplied CCXXVIII by XXXIV to get MMMMMMMDCCLII has the same right answer as the modern schoolchild who (correctly) multiplies 228 by 34 to get 7752. The difference is in the burden each method places on the human mind. This cannot be measured exactly but is related to such things as the time it takes to do typical sums, the time needed to learn the method, the effort required of the memory and the care needed to prevent mistakes. In some cases one method may be virtually unable to cope with some types of problem: how, for instance, would you handle fractions or divisions in Roman numerals?

The claim I make is that the burden of arithmetic in my life would have been at least halved if we used, and I had been taught, the system I am about to describe. I shall give enough information for anyone with O-level aptitude to be able to go away and fill in the remaining details. If you are prepared to spend a few hours practising the method you will be able to form your own judgement of its virtues and see whether you agree with me. I would be interested to hear any reactions, especially from schools.

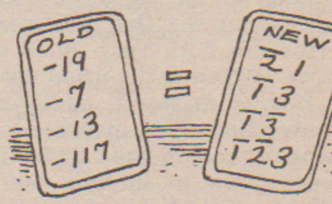
The starting point for arithmetic is counting. Suppose that we learnt to count as follows: one, two, three, four, five, ten-less-four, ten-less-three, ten-less-two, ten-less-one, ten, ten-and-one, ten-and-two, . . . and so on, with the same pattern being used for higher numbers. There is nothing strange about this way of counting: we use it every day when we tell the time. We say "twenty-five to three" rather than "thirty-five minutes past two" and "three minutes to four" rather than "fifty-seven minutes past three". The spirit of this method is that the time (or number) is first estimated to the nearest hour (or ten) and then extra minutes (or units) are added or subtracted to give the exact result.

Naturally, if our ancestors had used such a method of counting they would have abbreviated the names they used, rather as four-and-ten has become fourteen and four-tens has become forty. Thirty-less-two might have become "thirlytwo", and we would have counted ". . . thirlytwo, thirlyone, thirty, thirtyone, thirtytwo, . . .".

But a choice of abbreviated names is far less important than the way we write the numbers. The best way of writing numbers following the new philosophy seems to be: 1, 2, 3, 4, 5, $1\bar{4}$, $1\bar{3}$, $1\bar{2}$, $1\bar{1}$, 10, 11, 12, 13, 14, 15, $2\bar{4}$, $2\bar{3}$, . . . Notice that the numbers 1 to 5, 10 to 15, and so on, are written normally. The number 8, however, being thought of as ten-less-two, or ten minus two, is written $1\bar{2}$. The bar over the 2 can be thought of as a minus sign, showing that we mean 10 minus 2 and not 10 plus 2. The *position* of a digit shows whether it is a unit, a ten or a hundred in the usual way; a bar (or negative sign) above a digit shows that it represents a negative number of units, tens or hundreds. Thus $2\bar{3}4$ means 2 hundreds and −3 tens and +4 units, which is to say 200−30+4=174 in modern notation. Decimal numbers are included naturally. For example, $3\bar{2}\cdot\bar{4}$ has +3 tens, −2 units and −4 tenths, and so is the same as 30−2−0·4=27·6 in modern notation; likewise $2\bar{1}\cdot\bar{3}\bar{1}$=20−1−0·3−0·01=18·69.



Negative numbers are implicit in the system, because by definition $\bar{2}$ means −2. A number such as $\bar{2}3$ can be translated using the standard rule to give −2 tens and +3 units or −17, which is also negative. In the new notation the overall sign of a number, whether it is positive or negative, is the same as the sign of the leftmost digit. The rule for writing down the negative of a given number is simple. *The sign of each digit is changed.* Thus each unbarred (positive) digit is barred (made negative) and vice versa. For example the number 19 is $2\bar{1}$ in the new notation and so −19 becomes $\bar{2}1$.



By this stage many readers will have noticed an oddity about the number 5: it can also be written consistently as $1\bar{5}$, or 1 ten less 5 units. This is no more strange than the fact that "thirty minutes past two" is the same as "thirty minutes to three", and leads to no real problems. The worst that can be said about it is that it is an unaesthetic feature of the system. If it upsets you a lot then one remedy is to

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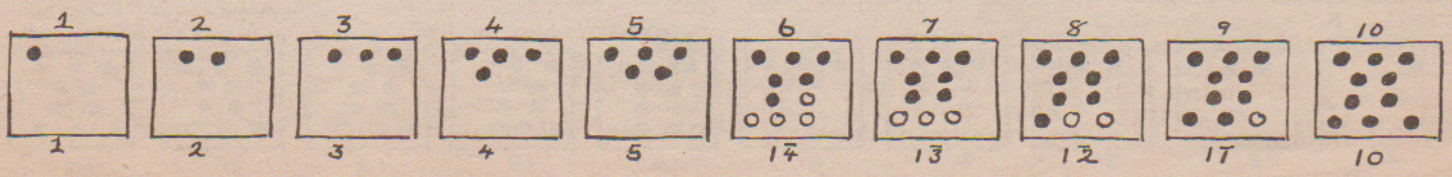
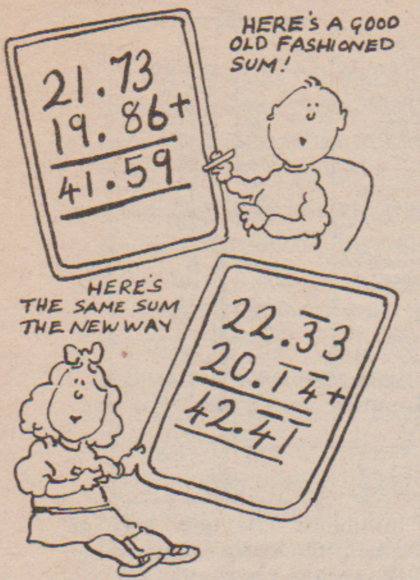


go for an out-and-out reform and count in multiples of an odd number like eleven in place of ten; the unattractive feature then disappears. But in this article I will retain the base of ten because it is more familiar.

Now I have explained how to count and to write numbers, I shall describe how to do some calculations, beginning with addition and subtraction. The first step is to memorise the sums and differences of the digits. Children nowadays have to learn that $7+8=15$ and $-9+4=-5$, for example. In the new system this basic knowledge is much more easily acquired, because it involves only mastery of the sums and differences of the digits 1 to 5. Roughly speaking, we have all learnt three times as much as we would have needed under the new system; our only problem now is translating what we know into the new notation. For example, $3+4=7$ becomes $3+4=1\bar{3}$. Similarly $3+(-5)=-2$ becomes $3+\bar{5}=\bar{2}$ and so on. The ability to think of numbers in the new notation rather than having to translate from the old notation comes with practice.

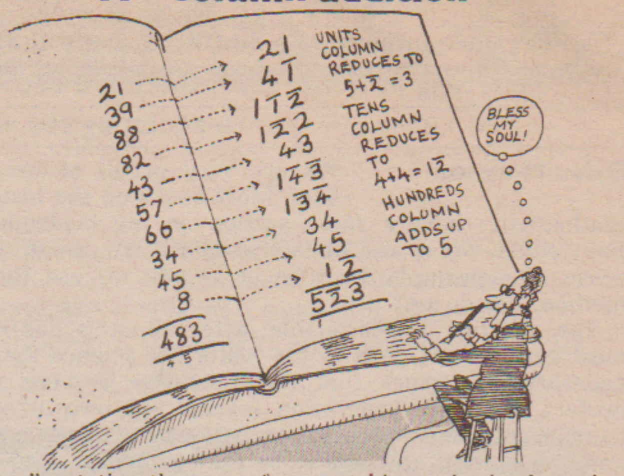


Consider next the exercise that has consumed most of mankind's mathematical effort: adding columns of figures. It is awe-inspiring to think of the armies of clerks who have worked their lives away in the task of adding and balancing accounts. Yet their task could have been made very much easier.

Imagine a typical ledger column of some 30 entries, written in the new way (Box A). Pencil in hand, look down the unit column, lightly deleting any matching pair, 1 and $\bar{1}$, 2 and $\bar{2}$, and so on, *because they add to zero*. What a soothing and pleasant task this is, totally undemanding and with no risk of error! If you are feeling a little brighter you will notice also that the digits 1, 1 and $\bar{2}$ can be deleted together, or $\bar{3}$, $\bar{2}$ and 5 and so on. Within a few seconds the column of figures is reduced to a fraction of its original size, and the few remaining digits can be added with little strain. The running total seldom gets very large because negatives and positives tend to cancel. Furthermore there are none of those troublesome 6s, 7s, 8s and 9s, which always need the most care. If the column's total lies outside the range $\bar{5}$ to 5 a digit is carried to the next column in the usual way, and the process is repeated. It is easy to see that the mental effort and the chance of error are both very much smaller than if the same sum were written in the usual notation.

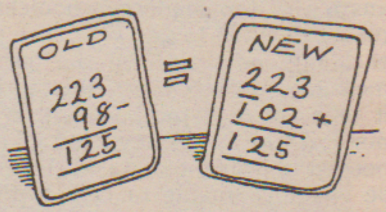
Having mastered addition, children move on to learn subtraction. This involves learning a completely different process, complicated by having to transfer quantities from one column to another nearly half the time. Think for a minute of the millions of man-years that have been devoted to learning and teaching subtraction. In contrast, subtraction in the new arithmetic uses exactly the same method as addition; there is no special algorithm to be learned. This is because an elementary step converts any subtraction into an addition. For example, $223-98$ is complicated in our present method because 8 is greater than 3, and 9 is greater than 2. In the new notation the problem becomes

A Column addition



In "bar" notation there are four matching pairs in the units column, which can be deleted, reducing it to $5+\bar{2}=3$. The tens column similarly reduces to $4+4=1\bar{2}$. The hundreds column is $1+1+1+1+1=5$. Thus only seven simple additions are required compared with the 18 elementary additions in contemporary notation.

$223-10\bar{2}$. But the rule for taking the negative of a number changes this at once to the *addition* sum $223+\bar{1}02$. Then $3+2=5$, $2+0=2$, $2+\bar{1}=1$ and the answer is 125.



Next consider a column of numbers, some of which are positive and some negative. This arises naturally in a financial situation where credits are positive and debits negative. Equally it might arise in a series of temperature readings that have to be averaged. In the notation we use normally it is impossible to sum such a column directly. The positive and negative entries must first be separated, added, and the sums subtracted from each other. But in the new notation the column can be summed as it stands because positive and negative numbers are handled in the same way.

The next calculation children learn is multiplication. In this case, the necessary preliminary is the memorisation of the multiplication tables, a task I still recall with anguished clarity. If I had been taught using the new system I would have needed only the ten results involved in learning five short times-tables (see Figure 1), and my childhood would have been significantly happier. 7×8, for example, becomes $1\bar{3}\times1\bar{2}$ in the new notation, which involves only the same basic times-tables, although with "barred" values. The results of multiplying "barred" or negative digits follow at once from our normal rules for multiplying with negative numbers: remember that "minus times minus gives plus" so that $-(\bar{1})=1$, and so on. Long multiplication then gives the result of 7×8 as $1\bar{3}\times1\bar{2}=1\bar{4}\bar{4}=56$.

NEGATIVE MULTIPLICATION

×	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$
2	$\bar{4}$			
3	$\bar{1}4$	$\bar{1}1$		
4	$\bar{1}2$	$\bar{1}\bar{2}$	$\bar{2}4$	
5	$\bar{1}0$	$\bar{1}\bar{5}$	$\bar{2}0$	$\bar{2}5$

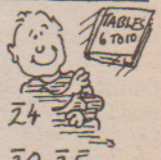
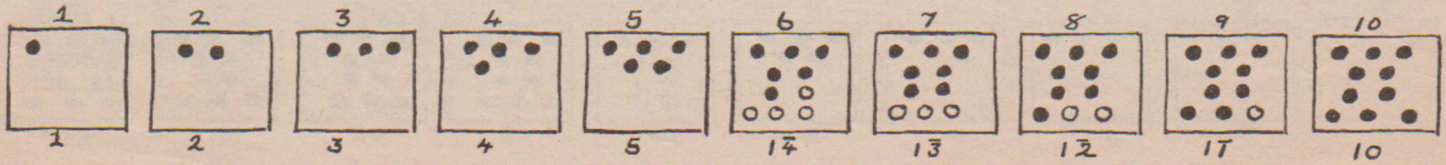


Figure 1 *The ten results needed for multiplication; it follows that* $\bar{4}\times3=4\times\bar{3}=\bar{1}\bar{2}$, $\bar{4}\times\bar{3}=12$ *and so on*



This simplification of the multiplication table is really enormous, as in practice it is the 6-, 7-, 8- and 9-times tables that provide nearly all the difficulty. It is because of this that long multiplication is made much easier with the new notation, although the increased simplicity of addition helps a little also.

Division is a process of repeated multiplications and subtractions, both of which are now made simpler. However, you will probably find it harder than for the other sums to change to doing division in the new notation. This is because at each step we must find *the closest* multiple of the divisor to the partial dividend and not *the nearest below*. For example, in dividing 140 by 5 write down 3 first rather than 2 first because $3\times5=15$ is *closer* to 14 than $2\times5=10$. The difference between 14 and 15 is -1 and so we "carry" $\bar{1}$. Then $\bar{1}0$ divided by 5 is $\bar{2}$, giving the overall answer $3\bar{2}$ (=28).

There is a further simplification that is useful if the division sum involves large numbers or if the answer is to have more than three or four significant figures. The trick is first to write out the multiplication table of the divisor, from 2 to 5. This can be done quite quickly by repeated addition (Box B). There is then no need to do any multiplication during the course of the calculation and it is less trouble deciding which is the correct multiple at each step. We do not use this technique in our present number system as it would take too much effort to write out the multiples from 2 to 9.

Finally, a small but useful feature of the new notation is the ease with which numbers are "rounded off". In the present system rounding off to the nearest whole number involves increasing 23·6 to 24 but decreasing 23·4 to 23. In the new system the same result is achieved by neglecting everything after the decimal point, in other words truncation and rounding off are identical.

I shall now offer some suggestions to readers who are interested in exploring the theme beyond its basics. Bases other than 10 can be used in the same way. Base 3 is a nice, easy case, using only the symbols $\bar{1}$, 0 and 1. This case is to "bar-arithmetic" what binary notation is to normal arithmetic, but is more compact, reducing to 5 binary digits a number of 8 binary digits. It would be a natural notation for a new generation of computers comprising active elements with three states. For example 1 could be represented by a positive current or charge or magnetic field, $\bar{1}$ by a negative current, charge or field, and 0 by zero values of these quantities.

In another direction it might be asked if double-entry book-keeping evolved only because our present number system cannot handle debits and credits in the same column. How would book-keeping with the new notation look? I said earlier that the bar notation is the best way of writing down numbers that are counted in the new way. Is this true? Perhaps negative digits could be written upside down, or in red, or on a computer visual display as black on white instead of white on black, or . . . there must be many ways.

That ends the description of new arithmetic. At this stage $1\bar{1}$ out of 10 of you will probably look doubtful and say that if it is as good as all that why has it never been described before? The answer is that it has. The earliest account seems to have been in $2\bar{3}3\bar{4}$ when John Colson, later to be Lucasian Professor at Cambridge, presented the system to the Royal Society (Box C). Since then other people have from time to time either read earlier accounts or (like myself) rediscovered the method. The only thing against it is that too many people have learnt our present system.

The mathematical advantages (which are real) of a change of system would be totally outweighed by the social disadvantages of the change. It would be like our recent change to decimalisation, but would involve most of the world and enormous expense. I suspect that the latest time such a change could have been made was in the $2\bar{4}$th or $2\bar{3}$rd century when there were still comparatively few people using written numbers. The change was not made then, and we have since invested too much in our less efficient system; so we will retain it for as long as our present civilisation lasts. There is much food for thought in this simple observation. □

C The first time round

John Colson FRS (1680-1760) was educated at Lichfield Grammar School and Christ Church, Oxford, which he left without taking a degree. In 1713 he was elected a fellow of the Royal Society and in May 1739 was appointed Lucasian Professor of Mathematics at Cambridge. A contemporary remarks that "he was a plain, honest man, of great industry and assiduity, but the university was much disappointed in its expectations of a professor that was to give credit to it by his lectures."

It was in 1726 that his work "A Short Account of Negativo-affirmative Arithmetick" appeared in the *Philosophical Transactions of the Royal Society*. This opening paragraph is an excellent summary:

"The Usefulness of this Arithmetick consists in this, that it performs all the Operations with more Ease and Expedition than the common Affirmative Arithmetick, especially in large Numbers: And it differs from the common Arithmetick chiefly in this, that it admits of Negative Figures promiscuously with the affirmative. These negative Figures are distinguish'd from the Affirmative, by the Sign "-" placed over them." □

B Long division

Contemporary notation	'Bar' notation
541.946	$542.\bar{1}5\bar{3}$
432) 234121.0000	432) 234121.000
2160	$\times 2 = 1\bar{1}\bar{4}4$; $\bar{2}\bar{2}40$
1812	$\times 3 = 130\bar{4}$; $2\bar{2}12$
1728	$\times 4 = 2\bar{3}3\bar{2}$; $\bar{2}3\bar{3}2$
841	$\times 5 = 22\bar{4}0$; $1\bar{2}41$
432	$114\bar{4}$
4090	$\bar{2}\bar{3}0$
3888	432
2020	2020
1728	$\bar{2}\bar{2}40$
2920	$\bar{1}\bar{4}00$
2592	$130\bar{4}$
3280	$\bar{1}0\bar{4}$

In contemporary notation, the answer is still not correct to three decimal places as the final 6 will have to be rounded up to 7. In "bar" notation, you use the multiplication table for the divisor obtained by repeated addition. Note also that each subtraction has been changed to an addition—and that the answer is correct to three decimal places. Now try dividing 234121 by 999 and by $100\bar{1}$. □